UNIT-I

LEBESGUE MEASURE

Set function:- An extended real valued function defined on any class of sets is called a "set function".

Measure of 'E':- A set function 'm' that assigns to each set 'E' in some collection '$\mathcal{M}$' of set of real numbers. A non-negative extended real number $m(E)$ is called the "Measure of E". which have the following properties

i, $m(E)$ is defined for each set E of real numbers ie, $\mathcal{M} = P(R)$

ii, for an interval 'I', measure of $I = l(I)$ ie, $m(I) = l(I)$

iii, if $\{E_n\}$ is a seq of disjoint sets then $m(\cup E_n) = \Sigma m(E_n)$

iv, ~~then~~ m is translation invarient ie, if E is a set for which 'm' is defined and if $E+y = \{x+y \mid x \in E\}$ obtain by replacing each point x in E by the point $x+y$ then $m(E+y) = m(E)$

Note:- 1, 'm' is called Additive if $\forall E_1, E_2 \in \mathcal{M} \Rightarrow$ measure of $E_1 \cup E_2 = m(E_1) + m(E_2)$ where $E_1 \cap E_2 = \phi$

2, m is called countabaly additive if measure of $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$

3, 'm' is called monotocity if $\forall E_1, E_2 \in m \ni E_1 \subseteq E_2 \Rightarrow m(E_1) \leq m(E_2)$

4, 'm' is called "Countabally sub-additive" if for each $E_i \in m$ we have $m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m(E_i)$

Length of an Interval:- Let 'I' be an interval in the real number system 'R' then the length of the interval is defined as the difference of two end points of the interval. It is denoted by $l(I)$

Outer measure of a set:- Let 'A' be any subset of 'R', a countable collection $\{I_n\}$ of open interval in 'R' is said to be a cover of 'A' if $A \subseteq \cup I_n$. consider the sum $\Sigma l(I_n)$ then the infimum of all sums taken over all covers of A is called the outer measure of 'A' and it is denoted by $m^*(A)$ ie, $m^*(A) = $ infimum $\{\Sigma l(I_n) \mid A \subseteq \cup I_n\}$ (or) $m^*(A) = \inf\{\Sigma l(I_n)$ where $A \subseteq \cup I_n\}$

Note:- If $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n,m} I_{nm}$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \Sigma \left(\underset{A\subset \cup I_n}{\ell(I_{nm})}\right)$$

Theorem:- i, $m^*(A) \geq 0 \ \forall A \subseteq R$

ii, $m^*(\phi) = 0$ as $n \to \infty$

iii, $m^*\{x\} = 0$ as $n \to \infty \ \forall x \in R$

iv, $A \subseteq B \Rightarrow m^*(A) \leq m^*(B)$

v, $m^*(A+x) = m^*(A)$ where $A+x = \{a+x \mid a \in A\}$ ie, outer measure is translation invarient

Proof:- Let $A \subseteq R$

By definition of outer measure ie, $m^*(A) = \inf\{\sum_n \ell(I_n) \mid A \subseteq \bigcup_n I_n\}$ where $\{I_n\}$ is a sequence of open intervals with $A \subseteq \bigcup_n I_n$ and $\ell(I_n)$ is length of an interval. we know that the length of an interval is non-negative ie, $\ell(I_n) \geq 0 \Rightarrow \sum_n \ell(I_n) \geq 0$

$$\Rightarrow \inf_{A\subset \cup_n I_n} \{\Sigma \ell(I_n)\} \geq 0$$

$$\Rightarrow m^*(A) \geq 0$$

$\therefore$ outer measure 'A' is non-negative.

ii, Let us consider an open interval $(\frac{-1}{n}, \frac{1}{n})$

Since $\phi \subseteq I_n = (\frac{-1}{n}, \frac{1}{n})$

Length of I_n ie, $\ell(I_n) = \ell(\frac{-1}{n}, \frac{1}{n})$

$$= \frac{1}{n} - (-\frac{1}{n}) = 2/n$$

By def of $m^*(A) = \inf\{\Sigma \ell(I_n) \mid A \subset \bigcup_n I_n\}$

$$m^*(\phi) = \inf\{\Sigma \ell(I_n) \mid \phi \subset \bigcup_n I_n\}$$

$$= \inf\{\Sigma(\tfrac{2}{n}) \mid \phi \subseteq \bigcup_n I_n\} \to 0 \text{ as } n \to \infty$$

$\therefore m^*(\phi) = 0$ as $n \to \infty$

iii, Let us consider $A = \{x\} \subseteq R$ also consider an interval $(x - \frac{1}{n}, x + \frac{1}{n})$ for $n \in Z^+$

By definition, $m^*(A) = \inf\{\Sigma \ell(I_n) \mid A \subset \bigcup_n I_n\}$

$$m^*\{x\} = \inf_{\{x\} \subset \cup_n I_n}\{\Sigma \ell(I_n)\}$$

$$= \inf_{\{x\} \subset \cup_n I_n}\{\Sigma(2/n)\}$$

$$= \bigcup_n I_n \to 0 \text{ as } n \to \infty$$

$\therefore m^*\{x\} = 0$ as $n\to\infty$ $\forall x \in R$

iv, Now we have to prove that $m^*(A) \le m^*(B)$

Let us take $A, B \in R$ and $A \subseteq B$

by definition $m^*(A) = \inf\limits_{\{I_n\}\in F} \{\sum_n l(I_n)\}$ where $F = \{\{I_n\} \mid A \subseteq \bigcup_n I_n\}$ —(1)

$m^*(B) = \inf\limits_{\{J_n\}\in S} \{\sum_n l(J_n)\}$ where $S = \{\{J_n\} / B \subseteq \bigcup_n J_n\}$ —(2)

where $\{I_n\}$ and $\{J_n\}$ are sequencies of open intervals with $A \subseteq \bigcup_n$

and $B \subseteq \bigcup_n J_n$

by eqn (2) $B \subseteq \bigcup_n J_n$

$\Rightarrow A \subseteq B \subseteq \bigcup_n J_n$ ($\because A \subseteq B$)

$\Rightarrow A \subseteq \bigcup_n J_n$

ie, $\{J_n\} \in S$

$\Rightarrow \{J_n\} \in F$

ie, $S \subseteq F \Rightarrow \inf F \le \inf S$

$\Rightarrow \inf\limits_{A \subset \bigcup_n I_n} \{\sum l(I_n)\} \le \inf\limits_{B \subset \bigcup_n J_n} \{\sum l(J_n)\}$

$m^*(A) \le m^*(B)$

v. we have to prove that $m^*(A+x) = m^*(A)$ where $A+x = \{a+x \mid a \in A\}$

we know that $m^* A = \inf\limits_{A \subset \bigcup_n I_n} \sum l(I_n)$

now for any $\epsilon > 0$ $\exists$ a sequence $\{I_n\}$ of an open-intervals with $A \subseteq \bigcup_n I_n$

$\ni m^*(A+\epsilon) > \sum_n l(I_n)$ —(1)

Since we have $A \subseteq \bigcup_n I_n$

$\Rightarrow A+x \subseteq \bigcup_n I_n + x$

$\Rightarrow m^*(A+x) \le \sum_n l(I_n + x)$

$\le \sum_n l(I_n)$

$\le m^*(A)+\epsilon$ —(2)

we know that $m^*(B) = \inf\limits_{B \subseteq \bigcup_n I_n} \sum_n l(I_n)$

for any $\epsilon > 0$ $\exists$ an sequence $\{I_n\}$ of an open interval with $B \subseteq \bigcup_n I_n$

$\ni m^*(B+\epsilon) > \sum_n l(I_n)$ —(3)

Since we have $B \subseteq \bigcup_n I_n$

$\Rightarrow B-x \subseteq \bigcup_n I_n - x$

$\Rightarrow m^*(B-x) \leq \sum_n \ell(I_n - x)$

$\leq \sum_n \ell(I_n)$

$\leq m^*(B) + \epsilon$

$\therefore m^*(B-x) \leq m^*(B) + \epsilon$

put $B = A+x$ ie, $m^*(A+x-x) \leq m^*(A+x)$

$m^*(A) \leq m^*(A+x)$ —(4)

from (2) and (4)

$m^*(A) = m^*(A+x)$

Theorem: 2 The outer measure of an interval is its length.

proof:- We prove this theorem in three cases.

case.i, Let 'I' be a finite closed interval say $I = [a,b]$

we know that $[a,b] \subset (a-\epsilon/2, b+\epsilon/2)$ for any $\epsilon > 0$

Hence $(a-\epsilon/2, b+\epsilon/2)$ is a cover of $[a,b]$

Hence by def, of outer measure we get

$\rightarrow \ell(a,b) = b-a$

$m^*[a,b] \leq \ell(a-\epsilon/2\ b+\epsilon/2) = b-a+\epsilon$ (outer measure def)

$\because \epsilon$ is arbitrary we get $m^*[a,b] \leq b-a$ —(1)

Now we prove that $m^*[a,b] \geq b-a$

To prove this it is sufficient to prove that if $\{I_n\}$ is a countable collection of open interval then $\sum_n \ell(I_n) \geq b-a$

Let $\{I_n\}$ be a cover of $[a,b]$ then by Hein-Borel Theorem we have to cover $\{I_n\}$ must have a finite subcover say $J_1, J_2, J_3 - - - J_k$

ie, $[a,b] \subset \bigcup_{i=1}^{k} J_i$ —(2)

we have $a \in [a,b] \Rightarrow a \in \bigcup_{i=1}^{k} J_i$

$\Rightarrow a \in J_i$ for some 'i'

denote the interval J_i which contain 'a' as (a_1, b_1) then we have $a_1 < a < b_1$.

If $b < b_1$ then stop the process otherwise we have $b_1 \in [a,b]$

Hence $b_1 \in \bigcup_{i=1}^{k} J_i$ (by eq^n ②)

Then $b_1 \in \cup J_i$ for some i

Denote the J_i containing b_1 by (a_2, b_2) then we have $a_2 < b_1 < b_2$

Now if $b < b_2$ stop the process Otherwise continuing the process we get a finite number of steps say 'n' steps where

We get a seq. $(a_1, b_1)(a_2, b_2) \ldots (a_r, b_r)$ open interval $\ni [a, \ldots$ is contained in $(a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_r, b_r)$

Hence we get that $\sum l(I_n) \geq \sum_{i=1}^{k} l(J_i)$ ($\because$ by ③)

$$\geq \sum_{i=1}^{r} l(a_i, b_i)$$

$$= (b_1 - a_1) + (b_2 - a_2) + \cdots (b_r - a_r)$$

$$= (b_r - a_r) + (b_{r-1} - a_{r-1}) + \cdots + (b_1 - a_1)$$

$$= b_r - (a_r - b_{r-1}) - (a_{r-1} - b_{r-2}) - \cdots (a_2 - b_1) - a_1$$

$$\geq b_r - a_1 \quad (\because a_i < b_{i-1} < b_i \ \forall i = 1, 2, 3 \ldots r)$$

Hence $\sum l(I_n) \geq b - a$ — (3) $\qquad \geq b - a \ (\because b < b_r \ \& \ a_1 < a)$

Since eq^n (3) is true for any cover $\{I_n\}$ of $[a, b]$ we get that

$\inf \{\sum l(I_n) \mid [a, b] \subseteq \cup I_n\} \geq b - a$

ie, $m^*[a, b] \geq b - a$ — (4)

from (1) and (4) we get $m^*[a, b] = b - a$

case ii :- Let I be any finite interval then given $\epsilon > 0$ we can find a closed interval $J \ni J \subseteq I$ and $l(J) > l(I) - \epsilon$

Hence we get that $l(I) - \epsilon < l(J)$

$$= m^*(J) \ (\text{by case } i)$$

$$\leq m^*(J) \ (J \subseteq I)$$

$$\leq m^*(\bar{I}) \leq m(\bar{I})$$

$$= l(\bar{I}) \ (I \subset \bar{I})$$

$$= l(I)$$

we have $l(I) - \epsilon \leq m^*(I) \leq l(I)$

$\because \epsilon > 0$ arbitrary we get that $m^*(I) = l(I)$

case iii :- Let 'I' be an infinite interval then we know that for any given real number $\delta > 0 \ \exists$ a $[J] \ni J \subseteq I$ and $l(J) = \delta$

Hence we get $m^*(I) \geq m^*(J)$

$$= l(J) \ (\text{by case } i)$$

$$= \delta$$

$\because \delta > 0$ is arbitrary we get that $m^*(I) = \infty = l(J)$

Hence proved.

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Theorem:-3 Let $\{A_n\}$ be a countable collection of set of real numbers then $m^*(\cup A_n) \leq \sum m^*(A_n)$

Proof:- If $m^*\{A_n\}$ is infinite then the inequality holds trivially.

If $m^*\{A_n\}$ is finite then given $\epsilon > 0$ $\exists$ a countable collection $\{I_{n,i}\}_i$ of open intervals $\ni A_n \subset \bigcup_i I_{n,i}$ and

$\sum_i l(I_{n,i}) \leq m^*(A_n) + \epsilon/2^n$ —(1)

Now the collection $\{I_{n,i}\}_{n,i} = \bigcup_i \{I_{n,i}\}_i$ is countable of covers $\cup A_n$

Thus $m^*(\cup A_{n,i}) \leq \sum_{n,i} l(I_{n,i})$

$= \sum_n \sum_i l(I_{n,i})$

$< \sum_n m^*(A_n) + \epsilon/2^n$ (by ①)

$\leq \sum m^*(A_n) + \epsilon$

$\therefore \epsilon$ is arbitrary we have $m^*(\cup A_n) \leq \sum m^*(A_n)$

Hence proved.

Corollary:4 If 'A' is countable then $m^*(A) = 0$

Proof:- Given that 'A' is countable set of real numbers.

By above theorem $m^*(A) \geq 0$

claim:- $m^*(A) = 0$

we know that, by definition the elements in a countable set can be arranged in a sequence of distinct terms

Let $A = \{x_n\}_{n=1}^{\infty} = (x_1, x_2, x_3 ---- x_n)$

ie, $A = \bigcup_{n=1}^{\infty} x_n$

$\Rightarrow m^*(A) = m^*(\bigcup_{n=1}^{\infty} x_n)$

$\leq \sum_{n=1}^{\infty} m^*(x_n)$

$= \sum_{n=1}^{\infty} m^*\{x_n\} \to 0$ as $n \to \infty$

$= 0$

$\Rightarrow m^*(A) \leq 0$

But, we know that $m^*(A) \geq 0$

$\therefore m^*(A) = 0$

Corollary:- The interval $[0,1]$ is not countable.

Proof:- Suppose it is possible $[0,1]$ is countable.

By above corollary, we have $m^*[0,1] = 0$

$\Rightarrow l([0,1]) = 0$

$l(a,b) = b-a$

$\Rightarrow 1 = 0$

which is a contradiction.

$\therefore$ Our assumption is wrong.

Hence $[0,1]$ is uncountable.

F_σ Set:- A set $F \subset R$ is said to be F_σ set if F is countable union of closed sets in R ie, $F = \bigcup_{i=1}^{\infty} F_i$

G_δ Set:- A set $G \subset R$ is said to be G_δ set if G is countable intersection of open sets in R ie, $G = \bigcap_{i=1}^{\infty} G_i$

Proposition:-5 Given any set 'A' any $\epsilon > 0$, there is an open set 'O' $\ni A \subset O$ and $m^*(O) \leq m^*(A) + \epsilon$ there is a 'G' and $G_\delta \ni A \subset G$ & $m^*(A) = m^*(G)$

Proof:- i, we know that, by definition of outer measure $m^*(A) = \inf\{l(I_n)\}$

where, $A \subset \bigcup_n I_n \Rightarrow \forall \epsilon > 0 \; \exists$ a sequence $\{I_n\}$ of open intervals with $A \subset \bigcup_n I_n \ni m^*(A) + \epsilon > \sum_n l(I_n)$ — (1)

Since we have $\{I_n\}$ is open interval in R.

So each $\{I_n\}$ is openset in R.

Let $\bigcup_n I_n = O$ be an openset in R.

Since, $A \subseteq \bigcup_n I_n = O$

$\therefore A \subset O$

Since $O = \bigcup_{n=1}^{\infty} I_n$

$$\Rightarrow m^*(O) = m^*\left(\bigcup_{n=1}^{\infty} I_n\right)$$
$$\leq \sum_{n=1}^{\infty} m^*(I_n)$$
$$= \sum_{n=1}^{\infty} l(I_n)$$
$$\leq m^*(A) + \epsilon$$

$\therefore m^*(O) \leq m^*(A) + \epsilon$

Now for any +ve integer 'N'

ii, consider an open set O_n in $R \ni A \subset O_n \; \forall n$

& $m^*(O_n) \leq m^*(A) + \frac{1}{n}$ — (2)

Since we have $A \subset O_n \; \forall n$

$\Rightarrow A \subset \bigcap_n O_n$ (G_δ set def)

Let $G = \bigcap_n O_n$

ie, G is a countable intersection of open sets 'R'

$\because G$ is a G_δ set $\ni A \subseteq \bigcap_n O_n$

$\Rightarrow A \subseteq G$

$\Rightarrow m^*(A) \leq m^*(G)$ —— (3)

$\therefore$ we have $G = \bigcap_n O_n$ i.e., $G \subseteq O_n$

$\Rightarrow m^*(G) \leq m^*(O_n)$

$< m^*(A) + \frac{1}{n}$

$\Rightarrow m^*(G) \leq m^*(A)$ $(\because \frac{1}{n} \to 0$ as $n \to \infty)$ — (4)

from (3) and (4)

$m^*(A) = m^*(G)$

Hence proved.

Problems:- 1. Prove that $m^*(A \cup B) = m^*(B)$ if $m^*(A) = 0$

Sol:- We know that $A \subset A \cup B$, $B \subset A \cup B$

$A \subset A \cup B \Rightarrow m^*(A) \leq m^*(A \cup B)$

$\Rightarrow 0 \leq m^*(A \cup B)$

$\Rightarrow m^*(A \cup B) \geq 0$ (because sub-additive property exists only $m^*(A \cup B) \geq 0$)

By countable sub additive property, we have

$m^*(A \cup B) \leq m^*(A) + m^*(B)$

$\Rightarrow m^*(A \cup B) \leq m^*(B)$

$B \subset A \cup B \Rightarrow m^*(B) \leq m^*(A \cup B)$

$\therefore m^*(A \cup B) = m^*(B)$

Measurable Sets And Lebesgue Measure

Definition:- A set E of a real numbers is said to be measurable if for any set $A \subset \mathbb{R}$ we have $m^*(A) = m^*(A \cap E) + m^*(A \cap \bar{E})$ where $\bar{E}$ is the complement of E in $\mathbb{R}$

$\bar{E} = \mathbb{R} - E$

Hence m^* is called Lebesgue outer measure.

Note:- For any set $A \subset \mathbb{R}$ we have $A = A \cap \mathbb{R}$

$= A \cap (E \cup \bar{E})$

$= (A \cap E) \cup (A \cap \bar{E})$

$m^*(A) \leq m^*(A \cap E) + m^*(A \cap \bar{E})$ — (1)

($\because$ By countable sub-additive property)

$\therefore$ we always have $m^*(A) \leq m^*(A \cap E) + m^*(A \cap \bar{E})$

we have that E is measurable iff for each 'A' we have

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \bar{E})$ — (2)

$\therefore$ from (1) and (2)

$\therefore m^*(A) = m^*(A \cap E) + m^*(A \cap \bar{E})$

Lemma: 6 If $m^*(E)=0$ then E is measurable.

Given that $m^*(E)=0$

claim:- E is measurable.

Let 'A' be any set $\Rightarrow A\cap E \subset E$

$\Rightarrow m^*(A\cap E) \le m^*(E)$

$\Rightarrow m^*(A\cap E) \le 0$

we know that outer measure is always non-negative

$\Rightarrow m^*(A\cap E) \geq 0$

$\therefore m^*(A\cap E) = 0$

we know that $A = (A\cap(E\cup \bar{E}))$

$= (A\cap E)\cup(A\cap \bar{E})$

$m^*(A) = m^*[(A\cap E)\cup(A\cap\bar{E})]$

$m^*(A) \le m^*(A\cap E) + m^*(A\cap \bar{E})$ —(1)

Lemma we know that $A\cap\bar{E} \subset A$

$\Rightarrow m^*(A\cap\bar{E}) \le m^*(A)$

$\Rightarrow m^*(A\cap\bar{E}) + 0 \le m^*(A)$

$m^*(A\cap\bar{E}) + m^*(A\cap E) \le m^*(A)$

$\Rightarrow m^*(A) \geq m^*(A\cap E) + m^*(A\cap\bar{E})$ —(2)

From (1) and (2) we have

$m^*(A) = m^*(A\cap E) + m^*(A\cap \bar{E})$

Lemma: 7 If E_1 & E_2 are measurable so is $E_1\cup E_2$

Given that E_1 & E_2 are measurable.

claim:- $E_1\cup E_2$ is measurable.

It is sufficient to prove that for any set $A\subset K$

$\Rightarrow m^*(A) \geq m^*(A\cap(E_1\cup E_2)) + m^*(A\cap(\overline{E_1\cup E_2}))$

Let $E_1\cup E_2 = E_1\cup E_2 - E_1$

$\Rightarrow A\cap(E_1\cup E_2) = A\cap[E_1\cup(E_2-E_1)] \Rightarrow m^*(A\cap(E_1\cup E_2)) \le m^*(A\cap E_1) + m^*(A\cap E_2\cap\bar{E}_1)$

$\Rightarrow m^*(A\cap(E_1\cup E_2)) + m^*(A\cap(\overline{E_1\cup E_2})) \le m^*(A\cap E_1) + m^*(A\cap(E_2\cap\bar{E}_1)) + m^*(A\cap(\overline{E_1\cup E_2}))$

$\le m^*(A\cap E_1) + m^*(A\cap(E_2\cap\bar{E}_1)) + m^*(A\cap(\bar{E}_1\cap\bar{E}_2))$

$= m^*(A\cap E_1) + m^*((A\cap\bar{E}_1)\cap E_2) + m^*((A\cap\bar{E}_1)\cap\bar{E}_2)$

$\le m^*(A\cap E_1) + m^*(A\cap\bar{E}_1)$ ($\because E_2$ is measurable)

$\le m^*(A)$ ($\because E_1$ is measurable)

$\therefore m^*(A) \geq m^*(A\cap E_1\cup E_2) + m^*(A\cap(\overline{E_1\cup E_2}))$

$\therefore E_1\cup E_2$ is measurable.

Corollary: The family $\mathcal{M}$ measurable sets is an algebra of sets.

We know that the family $\mathcal{M}$ of measurable sets is an algebra.

i, If E is measurable $\Rightarrow \bar{E}$ is measurable.

ii, $E_1 E_2$ is measurable $\Rightarrow E_1 \cup E_2$ is measurable.

To prove i:- By def., of measurable set E we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \bar{E})$$

Now for any set $A \subset R$ replace E by $\bar{E}$

$$\therefore m^*(A) = m^*(A \cap \bar{E}) + m^*(A \cap \bar{\bar{E}}) \Rightarrow m^*(A) = m^*(A \cap \bar{E}) + m^*(A \cap E)$$

$\therefore \bar{E}$ is measurable.

Hence E is measurable $\Rightarrow \bar{E}$ is measurable.

for the proof of ② write the above lemma

Lemma: Let 'A' be any set and $E_1, E_2 \cdots E_n$ is a finite sequence of disjoint measurable sets then $m^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} m^*(A \cap E_i)$

Sol:- Given that 'A' is any set and $E_1, E_2, E_3 \cdots E_n$ are finite seq. of disjoint measurable set.

claim:- $m^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} m^*(A \cap E_i)$ —(1)

ie, $m^*(A \cap (E_1 \cup E_2 \cup \cdots E_n)) = m^*(A \cap E_1) + m^*(A \cap E_2) + \cdots + m^*(A \cap E_n)$

We prove eqn ① by induction method

Let $n=1$

L.H.S. $m^*(A \cap \bigcup_{i=1}^{1} E_i) = m^*(A \cap E_1)$

R.H.S. $\sum_{i=1}^{1} m^*(A \cap E_i) = m^*(A \cap E_1)$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ The result is true for $n=1$

Let $n=2$

L.H.S. $\Rightarrow m^*(A \cap (E_1 \cup E_2)) = m^*\{(A \cap E_1) \cup (A \cap E_2)\}$

$= m^*(A \cap E_1) + m^*(A \cap E_2)$

$=$ R.H.S.

The result is true for $n=2$.

Assume that the result is true for $n-1$

ie, $m^*(A \cap (\bigcup_{i=1}^{n-1} E_i)) = \sum_{i=1}^{n-1} m^*(A \cap E_i)$

L.H.S:- $m^*(A \cap (\bigcup_{i=1}^{n-1} E_i)) = m^*(A \cap (E_1 \cup E_2 \cup \cdots E_{n-1}))$

$= m^*((A \cap E_1) \cup (A \cap E_2) \cup \cdots (A \cap E_{n-1}))$

$= m^*(A \cap E_1) + m^*(A \cap E_2) + \cdots + m^*(A \cap E_n)$

[$\because (E_1 \cup E_2 \cup E_3 \cdots \cup E_n)$ are disjoint so $(A \cap E_1) \cap (A \cap E_2) \cdots \cap (A \cap E_n)$ is empty]

$\sum_{i=1} m^*(A\cap E_i)+m^*(A\cap E_n)$

$=\sum_{i=1}^{n} m^*(A\cap E_i)$

$=$ R.H.S.

$\therefore$ The result is true for $n=i$

Theorem:- 10 The collection $\mathcal{M}$ of measurable set is a σ algebra ie, the compliment of a measurable set is measurable the union of a countable collection of measurable set is measurable. moreover, Every set with outer measure '0' is measurable.

Sol:- Given that $\mathcal{M}$ is a collection of measurable subsets of real numbers.

σ algebra - F_σ set

claim:- we have to prove that $\mathcal{M}$ is σ algebra.

It is sufficient to prove that i. $\mathcal{M}$ is algebra.

ii. The union of collection of measurable sets is also measurable

ie, $\bigcup_{i=1}^{\infty} E_i$ is measurable.

To prove i, ie, i=1

write the proof of corolary 8

To prove ii, consider $\{E_i\}_{i=1}^{\infty}$ be a seq. of disjoint measurable sets in '$\mathbb{R}$'

Let $E=\bigcup_{i=1}^{\infty} E_i$ & $F_n=\bigcup_{i=1}^{n} E_i$

we know that $F_n \subset E$

$\bar{F_n} \supseteq \bar{E}$

$A\cap \bar{F_n} \supset A\cap\bar{E}$

$\Rightarrow m^*(A\cap\bar{F_n}) \geq m^*(A\cap\bar{E})$ —(1)

$\therefore$ we have $F_n=\bigcup_{i=1}^{n} E_i$ —(2)

and $\{E_i\}_{i=1}^{\infty}$ are measurable.

$\Rightarrow \bigcup_{i=1}^{n} E_i$ is measurable.

$\Rightarrow F_n$ is measurable.

By definition of measurable set we have $m^*(A)=m^*(A\cap F_n)+m^*(A\cap\bar{F_n})$

$m^*(A) \geq m^*(A\cap F_n)+m^*(A\cap\bar{E})$ $(\because F_n \subset E)$

$= m^*\left(A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right)+m^*(A\cap\bar{E})$

$=\sum_{i=1}^{n} m^*(A\cap E_i)+m^*(A\cap\bar{E})$

$=\lim_{n\to\infty}\sum_{i=1}^{n} m^*(A\cap E_i)+\lim_{n\to\infty} m^*(A\cap\bar{E})$

$=\sum_{i=1}^{\infty} m^*(A\cap E_i)+m^*(A\cap\bar{E})$

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$\geq m^*(A\cap E) + m^*(A\cap \bar{E})$

$m^*(A) \geq m^*(A\cap E) + m^*(A\cap \bar{E})$ —(3)

but we have for any set contained in R

$m^*(A) \leq m^*(A\cap E) + m^*(A\cap \bar{E})$ —(4)

From (3) & (4)

$m^*(A) = m^*(A\cap E) + m^*(A\cap \bar{E})$

$\Rightarrow E$ is measurable.

$\Rightarrow \bigcup_{i=1}^{\infty} E_i$ is measurable.

$\therefore \mathcal{M}$ is ~~measurable~~ σ-algebra

Lemma:-11 prove that (a,∞) is measurable.

Let $E = (a,\infty)$

$\bar{E} = (-\infty, a]$

we have to prove that E is measurable.

It is sufficient to prove that $m^*(A) \geq m^*(A\cap E) + m^*(A\cap \bar{E})$

$\Rightarrow m^*(A) \geq m^*(A\cap (a,\infty)) + m^*(A\cap (-\infty,a])$

we know that by def. of outer measure of 'A'.

$$m^*(A) = \inf_{A\subseteq \bigcup_n I_n} \sum_n l(I_n) \quad —(1)$$

Now for $\epsilon > 0$ $\exists$ a $\{I_n\}$ of open interval with $A \subset \bigcup_n I_n$ such that

$m^*(A) + \epsilon > \sum_n l(I_n)$ —(2)

Let $A_1 = A\cap E$; $A_2 = A\cap \bar{E}$

$\Rightarrow A_1 = A\cap (a,\infty)$; $A_2 = A\cap(-\infty, a]$

$I_n' = I_n \cap E$; $I_n'' = I_n\cap \bar{E}$ (we define)

$\Rightarrow I_n' = I_n \cap (a,\infty)$; $I_n'' = I_n \cap (-\infty, a]$

we know that $I_n = I_n \cap R$

$= I_n \cap (E\cup \bar{E})$

$= (I_n \cap E)\cup (I_n\cap \bar{E})$

$= I_n' \cup I_n''$

$\Rightarrow m^*(I_n) = m^*(I_n') + m^*(I_n'')$

$\Rightarrow l(I_n) = l(I_n') + l(I_n'')$ —(3) $(\because m^*(I_n) = l(I_n))$

we have $A \subset \bigcup I_n$ (we define. 2)

$\Rightarrow A\cap E \subset \bigcup I_n \cap E$

$\Rightarrow A\cap E \subset \bigcup (I_n \cap E)$

$\Rightarrow A_1 \subset \bigcup I_n' \Rightarrow m^*(A_1) \leq m^*(\bigcup I_n')$

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$\Rightarrow m^*(A_1) \leq \Sigma l(I_n') \quad \text{---}(4)$

$\text{lly } m^*(A_2) \leq \Sigma l(I_n'') \quad \text{---}(5)$

from (4) and (5)

$$m^*(A_1) + m^*(A_2) = \Sigma l(I_n') + \Sigma l(I_n'')$$
$$= \Sigma l(I_n) \quad (\because \text{from } (3))$$
$$\leq m^*(A) + \epsilon \quad (\because \text{from } (2))$$

$\Rightarrow m^*(A) \geq m^*(A_1) + m^*(A_2)$ ($\because \epsilon$ is arbitrary)

$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap \bar{E})$

$\Rightarrow m^*(A) \geq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a])$

$\therefore (a, \infty)$ is measurable.

Borel Algebra:- Let '$\mathbb{B}$' be the smallest σ-algebra generated by classes of all open sets of 'R' then $\mathbb{B}$ is called "Borel Algebra". The numbers in $\mathbb{B}$ are called "Borel sets".

Note:- 1. $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$

2. $([a, b]) = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$

3. $([-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - \frac{1}{n})$

4. $(a, \infty) = \bigcup_{n=1}^{\infty} (a + \frac{1}{n}, \infty)$

Definition:- Let (A) is any set and it is said to be as measurable set if $m^*(A) =$ measurable of A $= m(A)$

Theorem:-12 Every borel set is measurable in particular each open set and each closed set is measurable

Proof: Since the collection $\mathcal{M}$ of measurable sets is a σ-algebra we have $(-\infty, a]$ is measurable for each A.

Since $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - 1/n)$

we have $(-\infty, b)$ is measurable.

Hence each open interval $(a, b) = (-\infty, b) \cap (a, \infty)$ is measurable.

But each openset is the union of countable number of open intervals and so must be measurable.

Thus, $\mathcal{M}$ is σ-algebra containing the open sets and must therefore contain the family $\mathbb{B}$ of ~~the~~ borel sets.

$\because \mathbb{B}$ is the Smallest σ-algebra containing the open sets.

Proposition:- 13 Let $\{E_i\}$ be a set of measurable sets then measure of $(\cup E_i) \le \sum m(E_i)$ if the sets $\{E_n\}$ are pair of disjoints then measure of $m(\cup E_i) = \sum m(E_i)$

proof:- Given that $\{E_i\}$ be a sequence of measurable sets

we know that $m^*(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} m^*(E_i)$ — (1)

Since each E_i is measurable we have

$m^*(E_i) = m(E_i)\ \forall i$ — (2) By defn $m^*(E) = m(E)$

and $\bigcup_{i=1}^{\infty} E_i$ is also measurable.

$\Rightarrow m^*(\bigcup_{i=1}^{\infty} E_i) = m(\bigcup_{i=1}^{\infty} E_i)$ — (3)

Sub. (2) and (3) in eqn (1) we get

$\Rightarrow m(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} m(E_i)$ — (4)

we know that if 'A' is any set & $E_1, E_2, E_3 --- E_n$ are finite seq. of measurable sets then

$m^*(A \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(A \cap E_i)$ (Lemma:-9)

Take $A = R$ $\because R = (E \cup E')$

$\Rightarrow m^*(R \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(R \cap E_i)$

$\Rightarrow m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(E_i)$

Consider $E = \bigcup_{i=1}^{\infty} E_i$

Now $\bigcup_{i=1}^{n} E_i \subset E$

$\Rightarrow m^*(\bigcup_{i=1}^{n} E_i) \le m^*(E)$

$\Rightarrow m(\bigcup_{i=1}^{n} E_i) \le m(E)$ ($\because$ E is measurable)

$\Rightarrow m(E) \geq m(\bigcup_{i=1}^{n} E_i)$

$\Rightarrow m(E) \geq \sum_{i=1}^{n} m(E_i)$

$\Rightarrow m(E) \geq \lim_{n \to \infty} \sum_{i=1}^{n} m(E_i)$

$\Rightarrow m(E) \geq \sum_{i=1}^{\infty} m(E_i)$ — (5)

Since each E_i is pair by disjoint we have, $m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m(E_i)$ — (5)

$m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$ $\because$ (4) and (5)

Hence proved,

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Proposition:- 14 Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets ie, A sequence with $E_{n+1} \subset E_n$ for each 'n'. of Let $m(E_1)$ be finite then $\lim_{n\to\infty} m(E_n) = m(\bigcap_{i=1}^{\infty} E_i)$

Proof:- Given that $\{E_n\}$ be an infinite decreasing sequence of mea-sure.

ie, A sequence with $E_{n+1} \subset E_n \; \forall n$

$\Rightarrow E_n \supset E_{n+1} \; \forall n = 1, 2, 3 \dots$

$\Rightarrow E_1 \supset E_2 \supset \dots E_n \supset E_{n+1} \supset \dots$

And also given $m(E_1) < \infty$

we prove that $m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$

For this let us take $E = \bigcap_{i=1}^{\infty} E_i$ — (1)

Since we have $E_n \subset E_1 \; \forall n$

we have $m(E_n) \leq m(E_1) < \infty$

$\Rightarrow m(E_n) < \infty \; \forall n$

Now consider, we have $E_{n+1} \subset E_n$

$\Rightarrow E_n = E_{n+1} \cup (E_n - E_{n+1})$

$\Rightarrow m(E_n) = m(E_{n+1}) \cup m(E_n - E_{n+1})$

$= m(E_{n+1}) + m(E_n - E_{n+1})$

$m(E_n - E_{n+1}) = m(E_n) - m(E_{n+1})$ —(2)

Let $F_i = E_i - E_{i+1}$ then $\bigcup_{i=1}^{\infty} F_i = E_1 - E$

$m(\bigcup_{i=1}^{\infty} F_i) = m(E_1 - E)$

$\Rightarrow m(E_1 - E) = m(\bigcup_{i=1}^{\infty} F_i)$

$\Rightarrow m(E_1 - E) = \sum_{i=1}^{\infty} m(F_i)$

$= \sum_{i=1}^{\infty} m(E_i - E_{i+1})$

$= \sum_{i=1}^{\infty} m(E_i) - \sum_{i=1}^{\infty} m(E_{i+1})$

$= \lim_{n\to\infty} \sum_{i=1}^{n} (m(E_i) - m(E_{i+1}))$

$= \lim_{n\to\infty} [m(E_1) - m(E_2)] + [m(E_2) - m(E_3)] + \dots + [-m(E_{n+1}) + m(E_n)]$

$= \lim_{n\to\infty} m(E_1) - m(E_n)$

$\Rightarrow m(E_1 - E) = \lim_{n\to\infty} m(E_1) - \lim_{n\to\infty} m(E_n)$

$\Rightarrow \cancel{m(E_1)} - m(E) = \cancel{m(E_1)} - \lim_{n\to\infty} m(E_n)$

$\Rightarrow m(E) = \lim_{n\to\infty} m(E_n)$

$\Rightarrow m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$

Hence proved.

Little woods First principle:- Let 'E' be a given set then the following five statements are equivalent

1, E is measurable

2, Given $\epsilon>0$ there is an open set $O \supset E$ with $m^*(O-E) < \epsilon$

3, Given $\epsilon>0$ there is a closed set $F \subset E$ with $m^*(E-F) < \epsilon$

4, There is a G in G_δ with $E \subset G$, $m^*(G-E)=0$

5, There is a F in F_σ with $F \subset E$, $m^*(E-F)=0$ if $m^*(E)$ is finite

the above statements are equivalent to

6, Given $\epsilon>0$ there is a finite union U of open intervals $\ni$ $m^*(U \Delta E) < \epsilon$

Proof:- To prove 1, $\Rightarrow$ 2 :-

Assume 1 ie, E is measurable.

Since E is measurable, we have $m^*(E) = m(E)$

But $m^*(E) = \inf\limits_{E \subset \bigcup_n I_n} \left(\sum_n l(I_n)\right)$

Now for any $\epsilon>0$ $\exists$ $\{I_n\}$ of open intervals with $E \subset \bigcup_n I_n \ni$

$m^*(E)+\epsilon > \sum_n l(I_n)$ —— (1)

Let $O = \bigcup_n I_n$

ie, O is the countable union of open sets $\ni E \subset \bigcup_n I_n = O$

$\Rightarrow E \subset O$

$\Rightarrow O \supset E$

$\Rightarrow O = (O-E) \cup E$

$\Rightarrow m^*(O) = m^*(O-E) + m^*(E)$

$\Rightarrow$ ~~$m^*(O) = m^*(O-E) + m^*(E)$~~

$\Rightarrow m^*(O-E) = m^*(O) - m^*(E)$ —— (2)

$\because O = \bigcup_n I_n \Rightarrow m^*(O) = m^*\left(\bigcup_n I_n\right)$

$= \sum_n m^*(I_n) = \sum_n l(I_n)$

$< m^*(E) + \epsilon$

$\Rightarrow m^*(O) - m^*(E) < \epsilon$

$\Rightarrow m^*(O-E) < \epsilon$

To prove (3) ⇒ (4) :

Assume (3) ie, for a given $\epsilon > 0$ $\exists$ an open set $O \supseteq E$ with $m^*(O-E) < \epsilon$ —(3)

Since by eq^n (3) $\forall n$ $\exists$ an openset $O_n \supseteq E \ni m^*(O_n - E) < \frac{1}{n}$ —(A)

Now let $G = \bigcap_n O_n$

ie, G is the countable intersection of opensets.

ie, G is in G_δ set $\ni E \subset \bigcap_n O_n = G$

$\Rightarrow E \subset G$

$\Rightarrow E \subset G \subset O_n \; \forall n$

$\Rightarrow (G-E) \subset O_n - E \; \forall n$

$\Rightarrow m^*(G-E) \leq m^*(O_n - E) < 1/n$

$\Rightarrow m^*(G-E) \leq 1/n$

$\Rightarrow \lim_{n\to\infty} m^*(G-E) \leq \lim_{n\to\infty} 1/n$

$\Rightarrow m^*(G-E) \leq 0$ —(4)

But we have outer measure is always non-negative

$\Rightarrow m^*(G-E) \geq 0$ —(5)

from (4) & (5)

$m^*(G-E) = 0$

To prove (4) ⇒ (1) :-

Assume (4)

ie, There is a G in G_δ with $E \subset G$

$m^*(G-E) = 0$

we have to prove that E is measurable.

we have $\exists$ G in G_δ

ie, $G = \bigcap_n O_n$

Here each O_n is an openset

$\Rightarrow$ G is measurable.

$\therefore E = G - (G-E)$

Hence E is measurable.

To prove (1) ⇒ (3) :-

Assume (1) ie, E is measurable.

Since (1) ⇒ (2) we have if E is measurable the for $\epsilon > 0$ $\exists$ an open set

'O' contains $E \ni m^*(O-E) < \epsilon$ — (6)

Since E is measurable we have $\bar{E}$ is also measurable.

E is measurable, by eq^n ⑥ we have
for $\epsilon > 0$ $\exists$ open set 'O' contains $\bar{E} \ni m^*(O-\bar{E}) < \epsilon$ —— (7)

$\Rightarrow \bar{E} \subseteq O$

$\Rightarrow \bar{O} \subset E$

Let $F = \bar{O}$ & Let $F \subset E$

$\because F = \bar{O} \Rightarrow E - F = E - \bar{O}$

$\Rightarrow m^*(E-F) = m^*(E - \bar{O}) = m^*(O - \bar{E})$ $(\because a - b = \bar{b} - \bar{a})$

$< \epsilon$ (by eq^n ⑦)

$\therefore m^*(E-F) < \epsilon$

To prove (3) $\Rightarrow$ (5):-

Assume iii; ie, Given $\epsilon > 0$ there is a closed set $F \subset E$ with $m^*(E-F) < \epsilon$ —— (8)

Since by eq^n ⑧ for each 'n' $\exists$ a closed set $F_n \subset E$ with $m^*(E-F_n) < \frac{1}{n}$ —— (9)

Let $F = \bigcup_{n=1}^{\infty} F_n$

ie, F is countable union of closed sets.

ie, F is F_σ set with $F \subset E$

Since $F = \bigcup_{n=1}^{\infty} F_n \Rightarrow F \supset F_n \ \forall n$

$\Rightarrow -F \subset -F_n \ \forall n$

$\Rightarrow E - F \subset E - F_n \ \forall n$

$\Rightarrow m^*(E-F) \le m^*(E-F_n) \ \forall n$ (by eq^n ⑨)

$\Rightarrow m^*(E-F) < \frac{1}{n}$

$\therefore \lim_{n\to\infty} m^*(E-F) \le \lim_{n\to\infty} \frac{1}{n}$

$m^*(E-F) \le 0$ —— (10)

But we have outermeasure is always non-negative.

$\Rightarrow m^*(E-F) \ge 0$ —— (11)

From (10) & (11) we have

$m^*(E-F) = 0$

To prove (5) $\Rightarrow$ (1):-

Assume (5) ie, There is F in F_σ with $F \subseteq E$

$m^*(E-F) = 0$ —— (12)

By eq^n (12) we have $m_n^*(E-F) = 0$

$\Rightarrow (E-F)$ is measurable.

By eq^n (12) we have F is in F_σ set

ie, $F = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed set

$\Rightarrow F$ is measurable.

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$\Rightarrow E = E - F + F$ is measurable.
$\Rightarrow E$ is measurable.
To prove (1) $\Rightarrow$ (6):-
Assume (1) ie, E is measurable.
Given that $E \subseteq R$ with $m^*(E) < \infty$
we prove that for each $\epsilon > 0$ $\exists$ a finite union of disjoint open interval 'U' $\ni m^*(U \Delta E) < \epsilon$
But we know that if E is measurable then for each $\epsilon > 0$ $\exists$ an open set $O \supset E \ni m^*(O - E) < \epsilon$ —(13)
$\Rightarrow m^*(O-E) < \epsilon/2$
$m^*(O) - m^*(E) < \epsilon/2$
$m^*(O) < m^*(E) + \epsilon/2$ —(14)
$m^*(O) - m^*(E) < \epsilon/2$
$m^*(O-E) < \epsilon/2$ —(15)
By eq^n (13) we have 'O' is an openset in 'R'.
ie, 'O' can be expressed as disjoint union of countable collection of open intervals $\{I_n\}$ in R.
ie, $O = \bigcup_{n=1}^{\infty} I_n$ —— (16)
$m^*(O) = m^*(\bigcup_{n=1}^{\infty} I_n)$
$m^*(O) = \sum_{n=1}^{\infty} m^*(I_n)$
$\Rightarrow m^*(O) = \sum_{n=1}^{\infty} l(I_n)$
$\therefore \sum_{n=1}^{\infty} l(I_n) = m^*(O) < m^*(E) + \epsilon/2$ (by eq^n (14))
$\Rightarrow \sum_{n=1}^{\infty} l(I_n) < \infty + \epsilon/2$
ie, $\sum_{n=1}^{\infty} l(I_n) < \infty$
Now for any $\epsilon/2 > 0$ $\exists N > 0 \ni \sum_{n=N+1}^{\infty} l(I_n) < \epsilon/2$ —— (17)
and let $U = \bigcup \bar{I_n}$ is the finite union of disjoint open intervals in 'R'.
Now, consider $E \Delta U = (E-U) \cup (U-E) \subseteq (O-U) \cup (O-E)$
$m^*(E \Delta U) \le m^*(O-U) + m^*(O-E)$
$= m^*(\bigcup_{n=N+1}^{\infty} I_n) + m^*(O-E)$
$= \sum_{n=N+1}^{\infty} l(I_n) + m^*(O-E)$
$< \epsilon/2 + \epsilon/2$
$< \epsilon$
$\therefore m^*(E \Delta U) < \epsilon$

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To prove (6) => (1):-

Assume (6) ie, for each $\epsilon > 0$ $\exists$ a finite union of disjoint open intervals $I_1, I_2, I_3 \ldots I_n \ni m^*(E \Delta U) < \epsilon$

ie, $U = \bigcup_{n=1}^{m} I_n$

we prove that E is measurable.

It is sufficient to prove that for any set $A \subset R$ we have $m^*(A) \geq m^*(A \cap E) + m^*(A \cap \bar{E})$

So consider $A \subset R$ we have $(A \cap E) \subset R$

Since $U = \bigcup_{n=1}^{\infty} I_n$ we have U is measurable.

By definition, we have $m^*(A \cap E) = m^*((A \cap E) \cap U) + m^*((A \cap E) \cap \bar{U})$ —(18)

we have $A \subset R, E \subset R \Rightarrow A \cap \bar{E} \subset R$ and we have 'U' is measurable

By def, we have $m^*(A \cap \bar{E}) = m^*((A \cap \bar{E}) \cap U) + m^*((A \cap \bar{E}) \cap \bar{U})$ —(19)

From (18) + (19) we have

$$m^*(A \cap E) + m^*(A \cap \bar{E}) = m^*((A \cap E) \cap U) + m^*((A \cap \bar{E}) \cap \bar{U}) + m^*(A \cap \bar{E} \cap U) + m^*((A \cap \bar{E}) \cap \bar{U}) \quad —(20)$$

we know that $A \cap (E \cap U) \subset A \cap U$

$A \cap (E \cap \bar{U}) \subset E \cap \bar{U}$ —(A)

$A \cap (\bar{E} \cap U) \subset \bar{E} \cap U$ —(B)

$A \cap (\bar{E} \cap \bar{U}) \subset A \cap \bar{U}$

By applying outer measure we have

$$\left.\begin{aligned} m^*(A \cap (E \cap U)) &\leq m^*(A \cap U) \\ m^*(A \cap (E \cap \bar{U})) &\leq m^*(E \cap \bar{U}) \\ m^*(A \cap (\bar{E} \cap U)) &\leq m^*(\bar{E} \cap U) \\ m^*(A \cap (\bar{E} \cap \bar{U})) &\leq m^*(A \cap \bar{U}) \end{aligned}\right\} —(21)$$

Sub. (21) in (20) we get

$$\begin{aligned} m^*(A \cap E) + m^*(A \cap \bar{E}) &\leq m^*(A \cap U) + m^*(E \cap \bar{U}) + m^*(\bar{E} \cap U) + m^*(A \cap \bar{U}) \\ &\leq m^*(A \cap U) + m^*(A \cap \bar{U}) + m^*(E - U) + m^*(U - E) \\ &\leq m^*(A) + m^*(E \Delta U) \\ &\leq m^*(A) + \epsilon \end{aligned}$$

Since 'ϵ' is arbitrary

$\epsilon + m^*(A) \geq m^*(A \cap E) + m^*(A \cap \bar{E})$

$\therefore E$ is measurable.

Hence proved.

Problem:-

:, If E is measurable set then each translate $(E+y)$ (E) is also measurable

Sol:- Given that E is measurable

By definition, for any set $A \subset R$ we have $m^*(A) = m^*(A \cap E) + m^*(A \cap \bar{E})$ -(1)

we have $m^*(E+x) = m^*(E)$ (theorem ①, condition)

Take $E = A \cap E$ and $x = y$

we know that $(A \cap E) + y = (A+y) \cap (E+y)$ —(2)

$m^*(A \cap E)+y = m^*(A+y) \cap (E+y)$ (By above)

$m^*(A \cap E) = m^*(A+y) \cap (E+y)$

Similarly $m^*(A \cap \bar{E}) = m^*(A+y) \cap (\bar{E}+y)$

$(1) \Rightarrow m^*(A) = m^*(A+y) \cap (E+y) + m^*(A+y) \cap (\bar{E}+y)$

$= m^*(A \cap (E+y)) + m^*(A \cap (\overline{E+y}))$

$\therefore E+y$ is measurable set.

Sum module '1' of x & y:- If x & y are real numbers in $[0,1)$ then we define sum modulo 1 of x & y as $x+y$ if $x+y<1$ and $x+y-1$ if $x+y \geq 1$ $\begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$

we denote sum modulo '1' of x & y as $x \overset{\circ}{+} y$ as a sum modulo 1 of x & y is binary operation.

Translate module '1' of E:- If $E \subset [0,1)$ then we define a translate modulo '1' of E to be the set $E \overset{\circ}{+} y = \{z \mid z = x \overset{\circ}{+} y$ for some $x \in E\}$

Lemma:-1.6 Show that Lebesgue measure is invarient under translation modulo '1' (or) If $E \subset [0,1)$ is a measurable set then for each $y \in [0,1)$ the set $x \overset{\circ}{+} y$ is measurable.

Proof:- Given that $E \subset [0,1)$ is measurable set.

we have to prove that for each $y \in [0,1)$ the set $E \overset{\circ}{+} y$ is measurable. (where $x \in E$)

For this, Let us define $E_1 = E \cap [0, 1-y)$

$E_2 = E \cap [1-y, 1)$

Then $E_1 \cap E_2 = \phi$ and $E_1 \cup E_2 = E$

$\therefore$ we have $E_1 \cup E_2 = E$

$\Rightarrow (E_1 \overset{\circ}{+} y) \cup (E_2 \overset{\circ}{+} y) = E \overset{\circ}{+} y$ —(1)

Again we have $E_1 \cup E_2 = E$

$\Rightarrow m(E_1 \cup E_2) = m(E)$

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$\Rightarrow m(E_1) + m(E_2) = m(E)$

$\Rightarrow m(E) = m(E_1) + m(E_2)$

But we know that $E_1 \mathring{+} y = \{z \mid z = x \mathring{+} y$ for some $x \in E_1\}$

$E_1 \mathring{+} y = E_1 + y$

$= E_1 \mathring{+} y$

$[\because x \in E_1 \Rightarrow x \in \{E_1 \cap [0, 1-y)\}$

So $x \in [0, 1-y)$

$\Rightarrow x \geq 1-y$

$\Rightarrow x+y \geq 1$ then by definition, of sum modulo '1' of x & y. we say that

$x+y \geq 1$ (By defintion of sum module '1' of x & y)

we have $x \mathring{+} y = 1$

$\Rightarrow E_1 + y - 1$

$\therefore E_1 \mathring{+} y$ is measurable.

($\because$ we know that if E_1, E_2 are measurable then $E_1 \cup E_2$ is measurable)

$\Rightarrow m(E \mathring{+} y) = m[(E_1 \mathring{+} y) \cup (E_2 \mathring{+} y)]$ ($\because$ by ①)

$= m(E_1 \mathring{+} y) + m(E_2 \mathring{+} y)$ (By applying measu. def. & problem)

$= m(E_1) + m(E_2)$ $\quad m^*(E_1 \mathring{+} y) = m^*(E_1) = m(E_1)$

$\therefore m(E \mathring{+} y) = m(E_1) + m(E_2)$

$= m(E_1 \cup E_2)$

$= m(E)$

$\therefore m(E \mathring{+} y) = m(E)$

Hence proved.

MEASURABLE FUNCTIONS

Proposition:- 1.8

Let 'f' be an extended real valued function whose domain is measurable then the following statements are equivalent

i, For each real number 'α' the set $\{x \mid f(x) > \alpha\}$ is measurable

ii, For each real number 'α' the set $\{x \mid f(x) \geq \alpha\}$ is measurable

iii, For each real number 'α' the set $\{x \mid f(x) < \alpha\}$ is measurable.

iv, For each real number 'α' the set $\{x \mid f(x) \leq \alpha\}$ is measurable

Proof:- Let the domain of extended real valued functions is 'D'.

By data, 'D' is measurable.

Proof:- First we prove that i, => iv, :-

Assume i, ie, for each real number 'α' the set $\{x \mid f(x) > \alpha\}$ is measurable.

We know that the difference of two measurable sets is also measurable.

$\therefore$ We have $D - \{x \mid f(x) > \alpha\}$ is also measurable.

ie, $\{x \mid f(x) \leq \alpha\}$ is measurable.

$\therefore$ i, => iv

Similarly we can prove that iv, => i,

$\therefore$ i, <=> iv,

To prove ii <=> iii :-

Assume ii ie, $\{x \mid f(x) \geq \alpha\}$ is measurable.

We have to prove that $\{x \mid f(x) < \alpha\}$ is measurable.

For each 'α'

We have $D - \{x \mid f(x) \geq \alpha\}$ is measurable.

ie, $\{x \mid f(x) < \alpha\}$ is measurable.

To prove i => ii :-

Assume i, ie, for each real number 'α' the set $\{x \mid f(x) > \alpha\}$ is measurable.

We know that $\{x \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > \alpha - \frac{1}{n}\}$

We have $\{x \mid f(x) > \alpha\}$ is measurable.

=> $\{x \mid f(x) > \alpha - \frac{1}{n}\}$ is also measurable.

We know that the arbitrary intersection of measurable sets is also measurable.

$\bigcap_{n=1}^{\infty} \{x \mid f(x) > \alpha - \frac{1}{n}\}$ is measurable.

$\therefore \{x \mid f(x) \geq \alpha\}$ is measurable.

To prove ii => i, :-

Assume ii, ie, for each real number 'α' the set $\{x \mid f(x) \geq \alpha\}$ is measurable.

We know that $\{x \mid f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \mid f(x) \geq$ [illegible]

$\therefore$ i, <=> ii,

we have $\{x \mid f(x) > \alpha\}$ is measurable

$\Rightarrow \{x \mid f(x) \geq \alpha + \frac{1}{n}\}$ is also measurable.

we know that arbitrary union of measurable sets is also measurable.

$\therefore \bigcup_{n=1}^{\infty} \{x \mid f(x) \geq \alpha + \frac{1}{n}\}$ is measurable.

$\therefore \{x \mid f(x) > \alpha\}$ is measurable.

Hence proved.

Definition:- An extended real valued function 'f' is said to be measurable if its domain is measurable and it satisfies the one of the four statements of above proposition.

Note:- i, $\{x \mid f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > n\}$

ii, $\{x \mid f(x) = -\infty\} = \bigcap_{n=1}^{-\infty} \{x \mid f(x) < -n\}$

Result:- If 'f' is a measurable function. Then prove that for each real number 'α' the set $\{x | f(x) = \alpha\}$ is measurable.

sol:- Given that 'f' is measurable.

Let 'α' be an real number.

Since 'f' is measurable then by above theorem $\{x | f(x) \leq \alpha\}$ & $\{x | f(x) \geq \alpha\}$ are measurable.

$\Rightarrow \{x | f(x) \leq \alpha\} \cap \{x | f(x) \geq \alpha\}$ is also measurable.

$\Rightarrow \{x | f(x) = \alpha\} = \{x | f(x) \leq \alpha\} \cap \{x | f(x) \geq \alpha\}$

$\Rightarrow \{x | f(x) = \alpha\}$ is measurable.

Let $\alpha = \infty$ then $\{x | f(x) = \alpha\} = \{x | f(x) = \infty\}$

$= \bigcap_{n=1}^{\infty} \{x | f(x) > n\}$ (by note point)

$\because$ f is measurable we have $\{x | f(x) > n\}$ is measurable.

$\Rightarrow \bigcap_{n=1}^{\infty} \{x | f(x) > n\}$ is also measurable.

$\Rightarrow \{x | f(x) = \infty\}$ is also measurable.

Let $\alpha = -\infty$

Since 'f' is measurable we have $\{x | f(x) < -n\}$ is measurable.

$\Rightarrow \bigcap_{n=1}^{-\infty} \{x | f(x) < -n\}$ is also measurable.

$\Rightarrow \{x | f(x) = -\infty\}$ is also measurable.

$\therefore \{x | f(x) = \alpha\}$ is measurable.

Proposition:-19 Let 'c' be a constant and f, g are two measurable real valued functions defined on the same domain then the functions $f+c$, cf, $f+g$, $g-f$ & gf are also measurable.

Proof:- Given that 'f' and 'g' are two measurable real valued functions define on the same domain and 'c' is any constant.

Now we prove that $f+c$ is measurable.

we know that if f is measurable then the set $\{x | f(x) < \alpha\}$ is measurable for any real number 'α'.

Consider $\{x | (f+c)(x) < \alpha\} = \{x | f(x) + c < \alpha\}$

$= \{x | f(x) < \alpha - c\}$ is measurable for any real number 'α' and arbitrary constant 'c'

$\therefore \{x | (f+c)(x) < \alpha\}$ is measurable.

$\therefore f+c$ is measurable function.

cf is measurable function:-

we know that if 'f' is measurable then the set $\{x \mid f(x) < \alpha\}$ is measurable for any real number 'α'.

consider $\{x \mid (cf)(x) < \alpha\} = \{x \mid c f(x) < \alpha\}$

$= \{x \mid f(x) < \alpha/c\}$ where $c \neq 0$ ie, $c > 0$

$\therefore$ The set $\{x \mid f(x) < \alpha/c\}$ is measurable for any real number α/c

$\therefore \{x \mid (cf)(x) < \alpha\}$ is measurable.

$\therefore$ cf is also measurable function.

f+g is measurable:- Now we prove that for this consider the set $\{x \mid (f+g)(x) < \alpha\} = \{x \mid f(x)+g(x) < \alpha\}$

$= \{x \mid f(x) < \alpha - g(x)\}$ (or)

$= \{x \mid g(x) < \alpha - f(x)\}$

Since here $f(x)$ & $\alpha - g(x)$ are two real numbers.

$\Rightarrow \exists$ a rational number 'r' $\ni f(x) < r < \alpha - g(x)$

$\Rightarrow f(x) < r, \; g(x) < \alpha - r$

Hence $\{x \mid f+g(x) < \alpha\} = \bigcup_{r \in Q} \{x \mid f(x) < r\} \cap \{x \mid g(x) < \alpha - r\}$

Here the set $\{x \mid f(x) < r\}$ is measurable. Since 'f' is measurable and the set $\{x \mid g(x) < \alpha - r\}$ is measurable. Since 'g' is measurable we know that the intersection of two measurable sets is again measurable. And also the countable union of measurable sets is again measurable.

So, $\{x \mid (f+g)(x) < \alpha\}$ is measurable $\therefore$ f+g is measurable.

Now we prove that g-f is measurable:-

we know that if 'f' is measurable then cf is also measurable.

Take $c = -1$ be any constant.

So, if f is measurable $\Rightarrow -f$ is also measurable.

By data, g is measurable.

$\Rightarrow g + (-f)$ is also measurable.

$\Rightarrow g - f$ is also measurable.

Now we prove that fg is measurable function:-

To prove this, first we prove if f is measurable then f^2 is also measurable.

Now consider, $\{x \mid f^2(x) < \alpha\} = \{x \mid f^2(x) - \alpha < 0\}$

$= \{x \mid f^2(x) - (\sqrt{\alpha})^2 < 0\}$

$= \{x \mid (f(x) + \sqrt{\alpha})(f(x) - \sqrt{\alpha}) < 0\}$

$= \{x \mid (f(x) + \sqrt{\alpha}) < 0\} + \{x \mid f(x) - \sqrt{\alpha} > 0\}$

$= \{x \mid f(x) < -\sqrt{\alpha}\} \cap \{x \mid f(x) > \sqrt{\alpha}\}$

Here the sets $\{x \mid f(x) < -\sqrt{\alpha}\}$ & $\{x \mid f(x) > \sqrt{\alpha}\}$ are measurable.

because 'f' is measurable

Also, the intersection of two measurable sets is again measur-able

So, $\{x \mid f^2(x) < \alpha\}$ is measurable.

$\Rightarrow f^2$ is measurable

Thus if, f is measurable we have f^2 is also measurable.

Ily, if g is measurable we have g^2 is also measurable.

Ily, if $f+g$ & $f-g$ are measurable, we have

$(f+g)^2$ and $(f-g)^2$ are also measurable

W.K.T $fg = \dfrac{(f+g)^2 - (f-g)^2}{4}$

$\Rightarrow \dfrac{(f+g)^2 - (f-g)^2}{4}$ is measurable

$\Rightarrow fg$ is measurable

Definition:- If $\{x_n\}$ is define limit superior of x_n is $\overline{lt}\, x_n = \inf_n \sup_{k \geq n} x_k$ and limit interior of x_n as $\underline{lt}\, x_n = \sup_n \inf_{k \geq n} x_k$

Note:- 1, $\overline{lim}(-x_n) = -\underline{lim}\, x_n$

2, $\underline{lim}(-x_n) \leq \overline{lim}\, x_n$

3, If $\{x_n\}$ and $\{y_n\}$ are two sequences then $\underline{lim}\, x_n + \underline{lim}\, y_n \leq \overline{lim}\, x_n + \overline{lim}\, y_n$

4, $\inf_n f_n = \sup_n (-f_n)$

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Theorem:- 20 If f_n is sequence of measurable functions then the function $\sup\{f_1, f_2, f_3 - - - f_n\}$, $\inf\{f_1, f_2, f_3 - - - f_n\}$; $\sup_{n\geq 1} f_n$, $\inf_{n\geq 1} f_n$ and $\underline{\lim}\, f_n$ & $\overline{\lim}\, f_n$ of are all measurable functions.

Proof: Given that f_n is a sequence of measurable functions.

claim:- $\sup\{f_1, f_2, f_3 - - - f_n\}$, $\inf\{f_1, f_2, f_3 - - - f_n\}$; $\sup f_n$, $\inf f_n$, $\underline{\lim}\, f_n$ & $\overline{\lim}\, f_n$ are measurable.

Now first we prove that $\sup\{f_1, f_2, f_3 - - - f_n\}$ is measurable.

For this consider $\{f_1, f_2, f_3 - - - f_n\}$ is a finite collection of measurable functions.

Also let $h(x) = \sup\{f_1, f_2, f_3 - - - f_n\}$ and let α be any real number then we have $\{x \mid h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x \mid f_i(x) > \alpha\}$

$\because$ we have f_i, $1 \leq i \leq n$ are measurable

$\Rightarrow \{x \mid f_i(x) > \alpha\}$, $1 \leq i \leq n$ is measurable set.

$\Rightarrow \bigcup_{i=1}^{n} \{x \mid f_i(x) > \alpha\}$ is measurable set.

$\Rightarrow \{x \mid h(x) > \alpha\}$ is measurable.

$\Rightarrow h(x)$ is measurable function.

$\therefore \sup\{f_1(x), f_2(x), - - - - f_n(x)\}$ is measurable.

Now we prove that $\inf\{f_1, f_2, f_3 - - - f_n\}$ is measurable function

we know that $\inf_n f_n = \sup(-f_n)$ (Note iv)

ie, $\inf f_i = \sup(-f_i)$, $1 \leq i \leq n$

$\because$ we have f_i, $1 \leq i \leq n$ are measurable. By known theorem f_i, $1 \leq i \leq n$ is measurable and

By above theorem we have $\sup\{(-f_1), (-f_2), - - - (-f_n)\}$ is measurable function.

Since each f_i is measurable.

$\Rightarrow \sup(-f_i)$, $1 \leq i \leq n$ is measurable.

$\Rightarrow \inf f_i$ is measurable.

Now we prove that $\sup_{n\geq 1} f_n$ is a measurable function:-

For this, let $g(x) = \sup_{n\geq 1} f_n$

consider the set $\{x \mid g(x) < \alpha\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > \alpha\}$

Since by data, we have f_n's are measurable functions.

We have the ordinary union of measurable sets is also measurable.

We have $\bigcup_{n=1}^{\infty} \{x \mid f_n(x) > \alpha\}$ is measurable.

$\Rightarrow g(x)$ is a measurable function.

$\therefore \sup_{n \geq 1} f_n$ is measurable function.

Now we prove that $\inf_{n \geq 1} f_n$ is a measurable function:-

We know that $\inf_{n \geq 1} f_n = \sup_{n \geq 1} (-f_n)$

Since $-f$ is measurable whenever f is measurable.

we have $-f_i$ $(1 \leq i \leq n)$ also measurable whenever f_i is measurable. Then by above case, we have $\sup_{i \geq 1} (-f_i)$ is measurable.

$\Rightarrow \inf_{i \geq 1} f_i$ is measurable.

$\Rightarrow \inf_{n \geq 1} f_n$ is measurable.

Now prove that $\overline{\lim} f_n$ is measurable:-

we know that $\overline{\lim} f_n = \inf_n \left(\sup_{k \geq n} f_k \right)$

Here as above, we say that, $\sup_{k \geq n} f_k$ is measurable function &

also $\inf_n \left(\sup_{k \geq n} x_k \right)$ is a measurable function.

ie, $\lim f_n$ is measurable.

Hence proved.

Definition: Let 'f' be real valued functions we defined the positive part of 'f' as follows.

$$f^+(x) = f(x) \text{ if } f(x) = 0$$
$$= 0 \text{ if } f(x) < 0$$

The negative part of 'f' as follows.

$$f^-(x) = -f(x) \text{ if } f(x) \leq 0$$
$$= 0 \text{ if } f(x) > 0$$

Also we have $f^+(-x) = \max\{f(x), 0\}$ and

$$f^-(x) = \max\{-f(x), 0\}$$
$$= \min\{f(x), 0\}$$

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Note:- 1, $f = f^+ - f^-$

2, $|f| = f^+ + f^-$

3 If 'f' and 'g' are measurable functions defined on a set 'E' then $|f|$ is also a measurable function and $\max(f,g)$ and $\min(f,g)$ are also measurable functions.

Almost Everywhere:- A property 'p' said to hold ie, if the set of points where the property 'p' fails to hold is set of measure zero.

Note:- If f and g are extended real valued measurable functions with the same domain then f = g almost everywhere if $m(\{x \mid f(x) \neq g(x)\}) = 0$ and $f_n \to f$ almost everywhere if $m(\{x \mid \lim_{n\to\infty} f_n \neq f\}) = 0$

Proposition:- 21 If 'f' is measurable function ~~then~~ and f = g almost everywhere then g is measurable function.

Proof:- Given that 'f' is measurable function.

Then by known theorem, we have $\{x \mid f(x) > \alpha\}$ is measurable set. $\to (1)$

Also given that f = g almost everywhere. Then by definition,

$m(\{x \mid f(x) \neq g(x)\}) = 0$ —(2)

Let $E = \{x \mid f(x) \neq g(x)\} \Rightarrow m(E) = 0$ —(3)

Now we prove that g is measurable function.

consider for any real number 'α' $\{x \mid g(x) > \alpha\}$

$= \{x \mid f(x) > \alpha\} \cup \{x \mid g(x) > \alpha\} - \{x \mid g(x) \leq \alpha\}$ —(4)

$$\left.\begin{aligned} \text{Let } C &= \{x \mid g(x) > \alpha\} \subset E \\ D &= \{x \mid g(x) \leq \alpha\} \subset E \end{aligned}\right\} \text{—(5)}$$

$\because$ we have $C \subseteq E \Rightarrow m^*(C) \leq m^*(E) = 0$ $(\because m^*(E) = m(E))$

$\Rightarrow m^*(C) \leq 0$

But always outer measure is non-negative, we have

$\therefore m^*(C) \geq 0$

$\therefore m^*(C) = 0$

$\therefore$ C is a measurable set

$\Rightarrow \{x \mid g(x) > \alpha\}$ is measurable —(6)

Illy we have $D \subseteq E \Rightarrow m^*(D) \leq m^*(E) = 0$

$\Rightarrow m^*(D) \leq 0$

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But we have $m^*(D) \geq 0$

$\therefore m^*(D) = 0$

$\therefore D$ is a measurable set

$\Rightarrow \{x \mid g(x) \leq \alpha\}$ is measurable set — (7)

From (1), (4) & (6) we have 'g' is measurable function.

Hence proved.

First Version of Little Wood Second Principle

Statement:- Let 'f' be a measurable function defined on the interval $[a,b]$ and assume that 'f' takes the value $\pm\infty$ only on a set of measure zero then given $\epsilon > 0$ we can find a step function 'g' and continuous function 'h' $\ni |f-g| < \epsilon, |f-h| < \epsilon$ except on a set of measure $< \epsilon$

Proof:- we will prove this theorem, By the help of following lemma.

a, Given measure function 'f' on $[a,b]$ that takes the value $\pm\infty$ only on set of measure zero and given $\epsilon > 0$ there is a constant $m \ni |f| \leq m$ except on a set of measure $< \epsilon/3$

b, Let 'f' be measurable function on $[a,b]$. &

Given $\epsilon > 0$ and a constant 'm' there is a simple function $\phi \ni$ $|f(x) - \phi(x)| < m$ except $|f(x)| < m$

c, Given a simple function ϕ on $[a,b]$.

There is a step function 'g' on $[a,b] \ni g(x) = \phi(x)$ except on a set of a measure less than $\epsilon/3$

d, Given a step function 'g' on $[a,b]$. There is a constant function $h \ni g(x) = h(x)$ except on a set of measure $< \epsilon/3$

By lemma (a) $\exists$ a constant 'm' and a measure set 'A_1', $\ni$ $|f(x)| \leq m$ $\forall x \notin A_1$, $m(A_1) < \epsilon/3$ — (1)

By lemma (c) $\exists$ a step function 'g' and a measure set $A_2 \ni$ $g(x) = \phi(x)$ $\forall x \notin A_2$ & $m(A_2) < \epsilon/3$ — (2)

By lemma (d) $\exists$ a ~~continuous~~ constant function 'h' and measure set $A_3 \ni g(x) = h(x)$ $\forall x \notin A_3$ & $m(A_3) < \epsilon/3$ — (3)

Now let, $x \notin A_1 \cap A_2$

$\Rightarrow x \notin A_1$ and $x \notin A_2$

Take $x \notin A_2 \Rightarrow \phi(x) = g(x)$

By lemma (b) $|f(x) - \phi(x)| < \epsilon$ become as $|f(x) - g(x)| < \epsilon$ — (4)

we know that $m(A_1 \cup A_2) \leq m(A_1) + m(A_2)$

$$\leq \epsilon/3 + \epsilon/3$$

$$< \epsilon$$

$$\therefore m(A_1 \cup A_2) < \epsilon$$

Now let $x \notin A_1 \cap A_2 \cap A_3 \Rightarrow x \notin A_1$ and $x \notin A_2$ and $x \notin A_3$

Take $x \notin A_3 \Rightarrow g(x) = h(x)$

$\therefore$ Eq^n (4) $\Rightarrow |f(x) - g(x)| < \epsilon$ becomes $|f(x) - h(x)| < \epsilon$

we know that $m(A_1 \cup A_2 \cup A_3) = m(A_1) + m(A_2) + m(A_3)$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$

$$< 3\epsilon/3 < \epsilon$$

$\therefore m(A_1 \cup A_2 \cup A_3) < \epsilon$

Hence proved.

Note:- 1, A step function is measurable.

2, Every step function is a simple function.

3, Every simple function is need not be step function.

Little Wood's Three principles

Definition:- A sequence $\{F_n\}$ are extended real valued functions defined on a set 'E' is said to converges to a function point wise if to each $x \in E$ we have $F_n(x) \to F(x)$ ie, given $x \in E$ and $\epsilon > 0$ there is a 'n' $\ni n \geq N$ we have $|F_n(x) - F(x)| < \epsilon$

Definition:- We say that $F_n \to F$ almost everywhere on E if there is a set $B \subseteq E$ such that $m(B) = 0$ and $F_n \to F$ on $E - B$

Note:- 1, Measure of $\bigcap_n E_n$ ie, $m(\bigcap_n E_n) = \lt m(E_n)$

2, if $\lt m(E_n) = 0$ then for a given $\delta > 0$ $\exists$ a +ve integer 'n' $\ni$ $m(E_n) < \delta$

3, if f and g are measurable functions defined on a measurable set 'E'

i, $|f|$ is measurable.

ii, max (f, g) & min (f, g) are measurable functions.

Proposition:- 23 One Version of Little wood's Third Principle

Statement:- Let 'E' be a meas. set of finite measure and $\{f_n\}$ is a Sequence of measurable functions defined on E. Let 'f' be a real valued function $\ni$ for each x in E we have $f_n(x) \to f(x)$ then given $\epsilon > 0$ and $\delta > 0$ there is a meas. Set $A \subseteq E$ with $m(A) < \delta$ an integer $n \ni \forall x \notin A$ and $n \geq N$

$\Rightarrow |f_n(x) - f(x)| < \epsilon$

Proof:- Given that 'E' is measurable set of finite measure with $m(E) < \infty$ —(1)

Also given that $\{f_n\}$ is a sequence of measurable functions defined on E and $f: E \to R$ is a real valued function and $f_n(x) \to f(x) \ \forall x \in E$

ie, $\lim_{n \to \infty} f_n(x) = f(x) \ \forall x \in E$

Now, let $G_n = \{x \in E \mid |f_n(x) - f(x)| \geq \epsilon$, for some $n \geq N\}$ —(2)

Since, we have f_n is measurable and $f = \lim f_n$ then 'f' is measurable.

$\Rightarrow f_n - f$ is measurable.

$\Rightarrow |f_n - f|$ is also measurable.

by eq^n (2) G_n is measurable set for each 'n'.

Now, let $E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E \mid |f_n(x) - f(x)| \geq \epsilon \ \forall n \geq N\}$ —(3)

$\because G_n$ is measurable for each n, we have $\bigcup_{n=N}^{\infty} G_n$ is also measurable for each 'n'

$\Rightarrow E_N$ is measurable for each 'n' —(4)

Now, $E_{N+1} = \bigcup_{n=N+1}^{\infty} G_n \subset \bigcup_{n=N}^{\infty} G_n = E_N$

ie, $E_{N+1} \subset E_N \ \forall N$

So E_N is an infinite decreasing sequence of measurable set 'E'.

Since we have $f_n(x) \to f(x) \ \forall x \in E$

By definition, we have for a given $x \in E$ and $\epsilon > 0 \ \exists$ a +ve integer $N \ni \forall n \geq N \ |f_n(x) - f(x)| < \epsilon$ —(5)

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Since we have $E_{N+1} \subset E_N$ and for each $x \in E$, there must be some E_N to which $x \notin E_N$

ie, $x \notin E_N \Rightarrow x \notin \bigcap_{N=1}^{\infty} E_N$

$\Rightarrow \bigcap_{N=1}^{\infty} E_N = \phi$

$\Rightarrow m\left(\bigcap_{N=1}^{\infty} E_N\right) = m(\phi)$

$\Rightarrow \lim_{n\to\infty} m(E_N) = 0$ (by proposition 1.14)

$\Rightarrow$ for a given $\delta > 0$ $\exists$ +ve integer $N \ni m(E_N) < \delta$

ie, $m\{x \mid |f_n(x) - f(x)| \geq \epsilon$ for some $n \geq N\} < \delta$

Now, let $A = E_N$

Now A is measurable subset of E

Since E_N is measurable set of each N and $m(A) < \delta$ and for a given $\epsilon > 0$ $\exists$ a +ve integer $N \ni \forall x \notin E_N$ and $n \geq N$

$|f_n(x) - f(x)| < \epsilon$

$\therefore$ for a given $\epsilon > 0$ $\exists$ a +ve integer $N \ni x \notin A$ and $n \geq N$

$|f_n(x) - f(x)| < \epsilon$

Hence proved.

Theorem:- 24 Let E be a measurable set of finite measurable & f_n be a sequence of measurable function that converges a real valued function 'f' almost everywhere on E then given $\epsilon > 0$ and $\delta > 0$ there is a set $A \subset E$ with $m(A) < \delta$ and +ve integer $N \ni$ $|f_n(x) - f(x)| < \epsilon \ \forall x \notin A$ & $n \geq N$.

Proof:- Given that $f_n \to f$ almost everywhere on E —(1) (By def)

$\Rightarrow \exists$ a measurable set $E_0 \subset E \ni m(E_0) = 0$ and $f_n \to f$ on $E - E_0$

By data, we have each f_n is a sequence of measurable function on E.

$\Rightarrow \underline{\lim} f_n$ and $\overline{\lim} f_n$ are also measurable functions on E

Since we have $f_n \to f$ almost everywhere on E

$\Rightarrow \lim_n f_n = f$ almost everywhere on E.

$\Rightarrow \underline{\lim} f_n = f$ almost everywhere on E.

$\Rightarrow \overline{\lim} f_n = f$ almost everywhere on E.

$\Rightarrow f$ is measurable

Since $f_n \to f$ on $E-E_0$

$\Rightarrow$ for a given $\epsilon > 0$ and $\delta > 0$ $\exists$ a measurable set $A_0 \subset E-E_0$ with $m(A_0) < \delta$ and an integer $N \ni \forall x \notin A_0$ and $n \geq N$

$|f_n(x) - f(x)| < \epsilon$ (by above theorem)

Now define $A = A_0 \cup E_0$

Since here A_0 & E_0 are measurable sets and 'A' is measurable set and $A \subset E$ $(\because A_0 \subset E-E_0 \Rightarrow A_0 \cup E_0 \subseteq E \Rightarrow A \subseteq E)$

Since we have $A = A_0 \cup E_0$

$$\Rightarrow m(A) = m(A_0 \cup E_0)$$
$$\leq m(A_0) + m(E_0)$$
$$< \delta + 0$$
$$< \delta$$

Since, we have $x \notin A_0$ & $m(E_0) = 0 \Rightarrow E_0 = \phi$

$\Rightarrow x \notin E_0$ then $x \notin A_0 \cup E_0$

$\Rightarrow x \notin A$

$\therefore \forall x \notin A$ & $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

Hence proved.

Definition:- A sequence $\{f_n\}$ of measurable function defined on a set E is said to converge to a function uniformly on E. If given $\epsilon > 0$ $\exists$ +ve integer N $|f_n(x) - f(x)| < \epsilon$

A stronger form of Little wood's third principle

(or)

Egoroff's Theorem

Statement:- If $\{f_n\}$ is a sequence of measurable function that converges to a real valued function 'f' on a measurable set 'E' of finite measure, then given $\eta > 0$ there is a subset $A \subset E$ where $m(A) < \eta \ni f_n \to f$ uniformly on $E-A$.

Proof:- Given that $\{f_n\}$ is a sequence of measurable function defined on a set E with $m(E) < \infty$

Also given that $f_n \to f$ on E (3rd principle)

$\Rightarrow$ for a given $\epsilon > 0$ and $\delta > 0$ $\exists$ a set $A \subset E$ with $m(A) < \delta$ and the +ve integer $N \ni \forall x \notin A$ & $n \geq N$

$|f_n(x) - f(x)| < \epsilon$ —— (1)

Let us define $\delta = \frac{\eta}{2^n} > 0$ and $A = A_n$

$\therefore$ Eq^n (1) $\Rightarrow$ for a given $\epsilon > 0$ & $\frac{\eta}{2^n} > 0$ $\exists$ a set $A_n \subset E$ with

$m(A_n) < \frac{\eta}{2^n}$ and the +ve integer $N \ni |f_n(x) - f(x)| < \epsilon$ —(2)

$\forall x \notin A_n$ & $n \geq N$

Now, let $A = \bigcup_n A_n$

$$m(A) = m\left(\bigcup_n A_n\right)$$
$$= \sum_n m(A_n)$$
$$< \sum_n \frac{\eta}{2^n} \text{ (by (2))}$$
$$= \eta\left(\sum_n \frac{1}{2^n}\right)$$
$$= \eta \cdot 1 = \eta$$
$$\therefore m(A) < \eta$$

$\therefore$ we have $A = \bigcup_n A_n$

Here A_n is measurable subset of E and

we know that measurable union of measurable sets is aga

measurable. So 'A' is a measurable subset of E.

Now we prove that $f_n \to f$ uniformly on $E-A$

So, consider $\epsilon > 0$ & $x \in E-A$

$\Rightarrow x \notin A$

$\Rightarrow x \notin \bigcup_n A_n$

$\Rightarrow x \notin A_n \ \forall n$

Now, for $\epsilon > 0$ choose an integer $N_p = (N) \ni$ by definition, $n \geq N$

becomes as $n \geq N_p$

$\therefore \forall x \notin A_n \ \forall n$ and $n \geq N_p$ then by eq^n (2) we have $|f_n(x) - f(x)|$.

Hence $f_n \to f$ uniformly on $E-A$

Hence proved.

characteristic function of A (ψ_A):-

Let $A \subset R$ define $\psi_A : R \to R$ by $\psi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$.

This function ψ_A is called characteristic function of 'A'.

Result:- show that characteristic function of every set is measurable & ψ_A is measurable $\Leftrightarrow$ A is measurable.

proof:- Suppose that ψ_A is measurable function.

Now we prove that A is measurable set

consider $A = \{x \mid \psi_A > \frac{1}{2}\}$

$\because$ we have ψ_A is measurable, by def, the set $\{x \mid \psi_A > \frac{1}{2}\}$ is measurable.

ie, A is measurable.

conversely, suppose that A is measurable.

we prove that ψ_A is measurable function.

consider $\{x \mid \psi_A > \alpha\} = A$ is measurable.

if $0 \le \alpha \le 1$

But we prove that for any real number 'α' the set $\{x \mid \psi_A > \alpha\}$ is measurable set

$\Rightarrow \psi_A$ is measurable.

Hence proved.

Simple function:- A real valued function 'ϕ' is called a "simple function" if it is measurable and assumes only a simple function and has the values $\alpha_1, \alpha_2, \alpha_3 --- \alpha_n$ then $\phi = \sum_{i=1}^{n} \alpha_i \psi_{A_i}$

where $A_i = \{x \mid \phi(x) = \alpha_i\}$

Result:- show that the sum and product of simple functions is simple.

proof:- Let 'ϕ' and 'ψ' be any two simple functions such that

$\phi = \sum_{i=1}^{n} \alpha_i \psi_{A_i}$; $\psi = \sum_{j=1}^{m} \beta_j \psi_{B_j}$

$\phi + \psi = \sum_{i=1}^{n} \alpha_i \psi_{A_i} + \sum_{j=1}^{m} \beta_j \psi_{B_j}$

Since here ϕ and ψ are simple functions.

By definition, we have both ψ and ϕ are measurable and assumes only a finite number of values.

we know that $f+g$ is measurable if f, g is measurable

$\therefore \phi + \psi$ is measurable and assumes a finite number of values

So $\phi + \psi$ is a simple function.

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Now $\phi\cdot\psi = \sum_{i=1}^{n}\alpha_i \psi_{A_i} \cdot \sum_{j=1}^{m}\beta_j \psi_{B_j}$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i \beta_j \psi_{A_i} \psi_{B_j}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i \beta_j \psi(A_i \cap B_j)$$

Since here 'ϕ' and 'ψ' are measurable and assumes only fi-
-te number of values.

we know that fg is measurable if f, g are measurable

$\therefore \phi\psi$ is measurable and assumes only a finite number of values.

So, $\phi\psi$ is a simple function.

Hence proved.

Result:- i, $\psi_{A\cap B} = \psi_A \cdot \psi_B$

ii, $\psi_{A\cup B} = \psi_A + \psi_B - \psi_{A\cap B}$

iii, $\psi_{A'} = 1-\psi_A$ (or) $\psi_{X-A} = 1-\psi_A$

Proof:- i, To prove $\psi_{A\cap B} = \psi_A \cdot \psi_B$

Let $x \in A\cap B \Rightarrow x\in A$ and $x \in B$

[But we know that $\psi_A(x) = \begin{cases} 1 & \text{if } x\in A \\ 0 & \text{if } x \notin A \end{cases}$]

Since we have $x\in A\cap B \Rightarrow \psi_{A\cap B} = 1$

$x\in A \Rightarrow \psi_A = 1$

$x \in B \Rightarrow \psi_B = 1$

L.H.S. $= \psi_{A\cap B} = 1$

R.H.S. $= \psi_A \cdot \psi_B = 1\cdot 1 = 1$

$\therefore$ L.H.S. = R.H.S.

Now let $x \notin A\cap B \Rightarrow x\notin A$ & $x \notin B$

$x\notin A\cap B \Rightarrow \psi_{A\cap B} = 0$

$x\notin A \Rightarrow \psi_A = 0$

$x \notin B \Rightarrow \psi_B = 0$

L.H.S. $= \psi_{A\cap B} = 0$

R.H.S. $= \psi_A \cdot \psi_B = 0\cdot 0$

$= 0$

$\therefore$ L.H.S. = R.H.S.

ii) To prove that $\psi_{A\cup B} = \psi_A + \psi_B - \psi_{A\cap B}$

Let $x \in A\cup B \Rightarrow x \in A$ $x \in B$

Let $x \in A\cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A\cup B \Rightarrow \psi_{A\cup B} = 1$

$x \in A \Rightarrow \psi_A = 1$

$x \in B \Rightarrow \psi_B = 1$

$x \in A\cap B \Rightarrow \psi_{A\cap B} = 1$

L.H.S $= \psi_{A\cup B} = 1$

R.H.S. $= \psi_A + \psi_B - \psi_{A\cap B}$

$= 1 + 1 - 1 = 1$

$\therefore$ L.H.S. = R.H.S.

Let $x \in A\cup B \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \notin A\cap B$

$\Rightarrow x \in A\cup B \Rightarrow \psi_{A\cup B} = 1$

$x \in A \Rightarrow \psi_A = 1$

$x \notin B \Rightarrow \psi_B = 0$

$x \notin A\cap B \Rightarrow \psi_{A\cap B} = 0$.

L.H.S. $= \psi_{A\cup B} = 1$

R.H.S. $= \psi_A + \psi_B - \psi_{A\cap B}$

$= 1 + 0 - 0 = 1$

$\therefore$ L.H.S. = R.H.S.

Let $x \in A\cup B \Rightarrow x \notin A$ and $x \in B$

$\Rightarrow x \notin A\cap B$

$x \in A\cup B \Rightarrow \psi_{A\cup B} = 1$

$x \notin A \Rightarrow \psi_A = 0$

$x \in B \Rightarrow \psi_B = 1$

$x \notin A\cap B \Rightarrow \psi_{A\cap B} = 0$

L.H.S $= \psi_{A\cup B}$

$= 1$

R.H.S. $= \psi_A + \psi_B - \psi_{A\cap B}$

$= 0 + 1 - 0 = 1$

$\therefore$ L.H.S. = R.H.S.

Let $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin A \cap B$

$x \notin A \cup B \Rightarrow \psi_{A \cup B} = 0$

$x \notin A \Rightarrow \psi_A = 0$

$x \notin B \Rightarrow \psi_B = 0$

$x \notin A \cap B \Rightarrow \psi_{A \cap B} = 0$

L.H.S. $= \psi_{A \cup B} = 0$

R.H.S. $= \psi_A + \psi_B - \psi_{A \cap B}$

$= 0 + 0 - 0 = 0$

$\therefore$ L.H.S. = R.H.S.

iii, To prove $\psi_{A'} = 1 - \psi_A$

Let $x \in A' \Rightarrow \psi_{A'} = 1$

$x \notin A \Rightarrow \psi_A = 0$

R.H.S. $= 1 - \psi_A = 1 - 0 = 1$

L.H.S. $= \psi_{A'} = 1$

$\therefore$ L.H.S. = R.H.S.

Let $x \notin A' \Rightarrow \psi_{A'} = 0$

$x \in A \Rightarrow \psi_A = 1$

L.H.S. $= \psi_{A'} = 0$

R.H.S. $= 1 - \psi_A = 1 - 1 = 0$

$\therefore$ L.H.S. = R.H.S.

Hence proved.